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AVERAGING DISTANCES IN REAL QUASIHYPERMETRIC BANACH SPACES OF FINITE DIMENSION

BY

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ABSTRACT

The average distance theorem of Gross implies that for each real N-dimensional Banach space ($N \geq 2$) there is a unique positive real number $r(E)$ with the following property: For each positive integer n and for all (not necessarily distinct) $x_1, x_2, \ldots, x_n$ in E with $\|x_1\| = \|x_2\| = \cdots = \|x_n\| = 1$, there exists an x in E with $\|x\| = 1$ such that

$$\frac{1}{n}\sum_{i=1}^{n} \|x_i - x\| = r(E).$$

The main result of this paper shows, that $r(E) \leq 2 - 1/N$ for each real N-dimensional Banach space E ($N \geq 2$) with the so-called quasihypermetric property (which is equivalent to E is L^1-embeddable). Moreover, equality holds if and only if E is isometrically isomorphic to $\mathbb{R}^N$ equipped with the usual 1-norm.

1. Introduction

In 1964, O. Gross published the following remarkable result:

THEOREM A (O. Gross): *Let (X, d) be a compact connected metric space. Then there is a unique positive real number $r(X, d)$ with the following property: For each positive integer n and for all (not necessarily distinct) $x_1, x_2, \ldots, x_n$ in X, there exists an x in X such that*

$$\frac{1}{n}\sum_{i=1}^{n} d(x_i, x) = r(X, d).$$

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For a proof of this Theorem see [11]. An excellent survey on this topic is given in [8].

Remark 1: (a) In the situation of Gross's Theorem we say that (X,d) has the average distance property with rendezvous number (or averaging distance constant) $r(X,d)$.

(b) $D(X,d)/2 \leq r(X,d) < D(X,d)$, where $D(X,d)$ denotes the diameter of X. For a proof see Theorem 2 in [11].

(c) Graham Elton first generalized Gross's Theorem in the following sense (for a proof see [8]): Let (X,d) be a compact connected metric space and $M^1(X)$ be the set of all regular Borel probability measures on X; then $r(X,d)$ is the unique positive real number with the following property: For each μ in $M^1(X)$ there exists some x in X such that

$$\int_X d(x,y)\,d\mu(y) = r(X,d).$$

Moreover, there are μ_0, ν_0 in $M^1(X)$ with

$$\int_X d(x,y)\,d\mu_0(y) \leq r(X,d) \leq \int_X d(x,y)\,d\nu_0(y),$$

for all x in X.

Now let E be a real n-dimensional Banach space ($n \geq 2$). Consider the compact connected metric space (S_E, d) where $S_E = \{x \in E, \|x\| = 1\}$ denotes the unit sphere of E and d is the norm induced metric on S_E.

In [28] the rendezvous number $r(E)$ of E is defined as

$$r(E) = r(S_E, d).$$

For example, in [20] Morris and Nickolas proved that

$$r(l^2(n)) = \frac{2^{n-1}\left[\Gamma\left(\frac{n}{2}\right)\right]^2}{\sqrt{\pi}\,\Gamma\left(\frac{2n-1}{2}\right)}, \quad \text{for all } n \geq 2$$

and in [28] it is shown that

$$r(l^1(n)) = 2 - \frac{1}{n}, \quad r(l^\infty(n)) = \frac{3}{2}, \text{ for all } n \geq 2,$$

where $l^p(n)$ denotes $\mathbb{R}^n$ equipped with the usual p-norm.

Recently, the concept of rendezvous numbers was generalized to Banach spaces of infinite dimension. See [3], [19], [28] and [30].

This paper deals with a question given in [28]: Let E be a real n-dimensional Banach space ($n \geq 2$). Is it true, that

$$r(E) \leq 2 - \frac{1}{n} \left(= r(l^1(n))\right) \quad \text{for all } n \geq 2?$$

Up to now the conjecture $r(E) \leq 2 - \frac{1}{n}$, $n \geq 2$ was proved in the following two cases:

First it was shown in [28] that $r(E) \leq \frac{3}{2}$ for all two dimensional real Banach spaces and second the inequality $r(E) \leq 2 - 1/n$ was established in [29] for all $n \geq 2$ under the assumption that E has a 1-unconditional basis. Moreover, in both cases it was proved that equality holds only in the case $E = l^1(n)$ up to isometric isomorphisms.

The main part of this paper establishes the inequality $r(E) \leq 2 - 1/n$ for another class of finite dimensional Banach spaces, namely the class of all real n-dimensional Banach spaces ($n \geq 2$) with the so-called quasihypermetric property.

Let us remark that quite recently a general upper bound for $r(E)$ was established in [3]:

$$r(E) \leq 2 - \frac{1}{2 + (n-1)2^{n+1}},$$

for all real n-dimensional Banach spaces E ($n \geq 2$). (For a proof see Proposition 7.3 in [3].)

The remaining part developes a general inequality concerning averaging distances in two dimensional real Banach spaces:

It will be shown that for each compact convex subset K of a two dimensional real Banach space the inequality

$$\frac{1}{N^2} \sum_{i,j=1}^{N} \|x_i - x_j\| \leq \frac{D(K)}{2} + \frac{p(K)}{16}$$

holds for all $N \geq 1$ and $x_1, x_2, \ldots, x_N$ in K, where $D(K)$ denotes the diameter of K and $p(K)$ denotes the Minkowski perimeter of K. (Note the connection to a result given by L. Fejes Tóth in [10] for K the Euclidean unit ball in $\mathbb{R}^2$.)

2. Basic background and notation

Let (X, d) be a compact connected metric space. The rendezvous number $r(X, d)$ is defined as in Chapter 1 of this paper. With $M^+(X)$ we denote the space of all finite non-negative regular Borel measures on X. The subspace $M^1(X)$ of $M^+(X)$ consists of all non-negative regular Borel measures on X with total mass one (the probability measures on X).

A metric space (X, d) is called quasihypermetric if

$$\sum_{i,j=1}^{n} c_i c_j d(x_i, x_j) \leq 0,$$

for all n in $\mathbb{N}, x_1, \ldots, x_n$ in X and all $c_1, \ldots, c_n$ in $\mathbb{R}$ with $c_1 + \cdots + c_n = 0$. A normed linear space $(X, \|.\|)$ is called quasihypermetric if the metric space (X, d), where d denotes the norm induced metric on X, is quasihypermetric. Recall the following examples for quasihypermetric spaces:

(1) The Euclidean space $\mathbb{R}^n$ for all $n \geq 1$.

(2) $\mathbb{R}^n$ equipped with the usual p-norm for $1 \leq p \leq 2$ and all $n \geq 1$.

(3) All two dimensional real Banach spaces.

(For a proof of (1) and (2) see [25], for (3) see [32] and Theorem B mentioned below.)

The spaces $\mathbb{R}^n$ equipped with the usual p-norm for $2 < p \leq \infty$ and all $n \geq 3$ are not quasihypermetric. (For a proof see [9] and Theorem B.)

There is a wide range of literature concerning equivalent properties to the quasihypermetric property in classical geometry and functional analysis. For example, see [5], [6], [9], [12], [13], [14], [15], [18], [22], [24], [25] and [32].

Collecting some of these results we have:

THEOREM B: *Let $n \geq 1$ and consider some norm $\|.\|$ on $\mathbb{R}^n$. The following assertions are equivalent:*

(1) *The n-dimensional real Banach space $(\mathbb{R}^n, \|.\|)$ is quasihypermetric.*

(2) *The n-dimensional real Banach space $(\mathbb{R}^n, \|.\|)$ is isometrically isomorphic to a subspace of $L^1[0,1]$. ($(\mathbb{R}^n, \|.\|)$ is L^1-embeddable.)*

(3) *The norm $\|.\|$ admits a so-called Levy representation:*

There is some μ in $M^+(\Omega_{n-1})$ on the Euclidean unit sphere Ω_{n-1} of $\mathbb{R}^n$ such that

$$\|x\| = \int_{\Omega_{n-1}} |(x|y)| d\mu(y), \quad \text{for all } x \text{ in } \mathbb{R}^n,$$

where $(.|.)$ denotes the usual inner product on $\mathbb{R}^n$.

Without loss of generality one can assume that the measure μ in $M^+(\Omega_{n-1})$ is even ($\mu(A) = \mu(-A)$ for all Borel subsets A of Ω_{n-1}, where $-A = \{x \in \Omega_{n-1},\ -x \in A\}$). Under this additional assumption μ is unique.

(4) *The closed unit ball of the dual space ($\{x \in \mathbb{R}^n,\ \|x\|' \leq 1\}$) is a zonoid. (A zonoid is the limit body, with respect to the Hausdorff metric, of a sequence of zonotopes, where a zonotope is defined as the Minkowski sum of a finite number of line segments.)*

For a proof combine Proposition 1, Corollary 1.1 and Corollary 1.3 in [25] with Corollary 2.6 and Corollary 6.2 in [5]. For uniqueness see formula (1.1) in [24] and Theorem 2.8 in [5].

Now let E be a real n-dimensional Banach space ($n \geq 2$). With

$$S_E = \{x \in E,\ \|x\| = 1\}$$

we denote the unit sphere of S. The rendezvous number $r(E)$ is defined as in Section 1 of this paper.

Remember a basis $a_1, \dots, a_n$ in E is called an Auerbach basis of E if

$$\max_{1 \leq i \leq n} |\alpha_i| \leq \|\alpha_1 a_1 + \cdots + \alpha_n a_n\| \leq |\alpha_1| + \cdots + |\alpha_n|,$$

for all $\alpha_1, \dots, \alpha_n$ in $\mathbb{R}$.

For $n \geq 1$ and $1 \leq p \leq \infty$ let $l^p(n) = (\mathbb{R}^n, \|.\|_p)$, where $\|.\|_p$ denotes the usual p-norm on $\mathbb{R}^n$.

Finally, the canonical basis of $\mathbb{R}^n$ is denoted by $e_1, \dots, e_n$ and $(.|.)$ is the usual inner product on $\mathbb{R}^n$.

3. The results

First we obtain

THEOREM 1: *Let E be a real quasihypermetric Banach space. Then for all $N \geq 1$ and elements $x_1, \dots, x_N$ in X we have*

$$\frac{1}{2N} \sum_{i=1}^{N} \|x - x_i\| + \|x + x_i\| \leq \frac{1}{N} \sum_{i=1}^{N} \|x_i\| + \frac{N-1}{N} \|x\|,$$

for all x in $P(x_1, \dots, x_N) \subseteq E$ defined as

$$P(x_1, \dots, x_N) = \left\{ \sum_{i=1}^{N} \alpha_i x_i,\ \text{ such that } \alpha_1, \dots, \alpha_N \in \mathbb{R} \text{ with } \max_{1 \leq i \leq N} |\alpha_i| \leq 1 \right\}.$$

From this we get

THEOREM 2: *Let E be a real n-dimensional quasihypermetric Banach space and let $a_1, \dots, a_n$ be an Auerbach basis of E. Then we have*

$$\frac{1}{2n} \sum_{i=1}^{n} \|x - a_i\| + \|x + a_i\| \leq 1 + \frac{n-1}{n} \|x\|,$$

for all x in E with $\|x\| \leq 1$.

Remark 2: The inequality established in Theorem 2 is sharp in the following sense:

For each E there is at least one x (for example $x = 0$) such that equality holds. For $E = l^1(n)$ and the canonical basis we have equality for all x with $\|x\|_1 \leq 1$ (see Lemma 2, (1) of this paper). Moreover, if we left the quasi-hypermetric property the result is false in general:

Let $E = l^\infty(3)$, $a_1 = (-1,1,1)$, $a_2 = (1,-1,1)$, $a_3 = (1,1,-1)$. It is easy to see, that a_1, a_2, a_3 forms an Auerbach basis of $l^\infty(3)$ but

$$\frac{1}{6}\sum_{i=1}^{3} \|x - a_i\| + \|x + a_i\| = 2 > 1 + \frac{2}{3}, \quad \text{for } x = (1,1,1).$$

Then we prove

THEOREM 3: *Let E be a real n-dimensional quasihypermetric Banach space ($n \geq 2$). Then we have*

$$r(E) \leq 2 - \frac{1}{n}$$

and equality holds if and only if E is isometrically isomorphic to $l^1(n)$.

Now let E be a two dimensional real Banach space. Recall the definition of the so-called Minkowski perimeter $p(K)$ of a compact convex subset K of E:

$$p(K) = \sup_P l(P),$$

where $l(P)$ denotes the sum of the lengths over all sides of a convex polygon P inscribed in K, all lengths measured in the norm induced metric.

The next result is a generalization of a Theorem of L. Fejes Tóth (see Theorem 1 in [10]) concerning averages of distances in the Euclidean plane. We have

THEOREM 4: *Let E be a two dimensional real Banach space and let K be a compact convex subset of E. Then for all $N \geq 1$ and $x_1, \ldots, x_N$ in K we have*

$$\frac{1}{N^2}\sum_{i,j=1}^{N} \|x_i - x_j\| \leq \frac{D(K)}{2} + \frac{p(K)}{16},$$

where $p(K)$ denotes the Minkowski perimeter and $D(K)$ the diameter of K.

Various papers deal with questions concerning another interesting constant involving averages of distances:

Let (X, d) be a compact metric space and define

$$M(X,d) = \sup_{N\geq 1, x_1,\ldots,x_N \in X} \frac{1}{N^2}\sum_{i.j=1}^{N} d(x_i, x_j).$$

For example, see [1], [2], [4], [7], [10], [16], [21], [23], [26] and [31].

As a simple consequence of Remark 1(c) and the fact that

$$M(X,d) = \sup_{\mu \in M^1(X)} \int_X \int_X d(x,y) d\mu(x) d\mu(y)$$

we have

$$r(X,d) \leq M(X,d)$$

for all compact connected metric spaces. (For details and the relation between these two constants in quasihypermetric compact connected spaces, see [16] and [31].)

Now in [28] it was shown that

$$r(E) \leq 1 + \frac{p(OE)}{16},$$

for all two dimensional real Banach spaces ($p(OE)$ is the Minkowski perimeter of the closed unit ball OE of E).

Applying Theorem 4 we obtain the stronger inequality:

COROLLARY 1: *Let E be a two dimensional real Banach space. Then we have*

$$M(S_E, \|.\|) \leq 1 + \frac{p(OE)}{16},$$

$p(OE)$ defined as above.

Finally, let us draw the reader's attention to an unsolved problem in two dimensional Euclidean distance geometry:

What is the value of

$$\gamma(l^2(2)) = \sup_{\substack{N \geq 1,\, x_1, \ldots, x_N \in \mathbb{R}^2 \\ \|x_i - x_j\|_2 \leq 1}} \frac{1}{N^2} \sum_{i,j=1}^{N} \|x_i - x_j\|_2 \ ?$$

(For more detailed information see, for example, [16].)

It is conjectured (see [26]) that $\gamma(l^2(2)) = M(T, \|.\|_2)$ where T denotes the Reuleaux triangle of diameter one (numerical calculations show that $M(T, \|.\|_2) \approx 0,6675277$).

In [21] the positive real constant k_2 is defined as

$$k_2 = \sup r(X, \|.\|_2),$$

where X ranges over all compact connected subsets of the Euclidean plane with diameter one.

The authors showed

$$0,6675276 \leq k_2 \leq 0,7182336.$$

We improve the given upper bound:

COROLLARY 2:

$$k_2 \leq \gamma(l^2(2)) \leq \frac{1}{2} + \frac{\pi}{16} \quad (\approx 0,6963495).$$

Finally, we note that if we replace the Euclidean distance by the 1-norm induced distance and ask for the analogous constant $\gamma(l^1(2))$, the problem turns out to be much easier:

COROLLARY 3: *Let E be a two dimensional real Banach space. Define*

$$\gamma(E) = \sup_{\substack{N\geq 1,\, x_1,\ldots,x_N\in E\\ \|x_i-x_j\|\leq 1}} \frac{1}{N^2}\sum_{i,j=1}^{N} \|x_i - x_j\|.$$

Then we have

(1) $\gamma(E) \leq \frac{3}{4}$,
(2) $\gamma(l^1(2)) = \frac{3}{4}$.

4. The proofs

We need the following

LEMMA 1: *Let $n \geq 1$ and $\alpha_1, \ldots, \alpha_n, \alpha$ in $\mathbb{R}$ with $\alpha \geq 0, \alpha_i \geq 0$, for all $1 \leq i \leq n$ and $\alpha \leq \alpha_1 + \cdots + \alpha_n$. Then we have*

$$\frac{1}{n}\sum_{i=1}^{n} \max(\alpha_i, \alpha) \leq \frac{1}{n}\sum_{i=1}^{n} \alpha_i + \frac{n-1}{n}\alpha.$$

Moreover, equality holds if and only if

1. $\alpha_1 = \cdots = \alpha_n = \alpha = 0$.
2. *There is some $1 \leq i_0 \leq n$, such that* $\alpha_{i_0} > 0$ *and* $\alpha_1 = \cdots = \alpha_{i_0-1} = \alpha_{i_0+1} = \cdots = \alpha_n = 0$.
3. *There are at least two indices $1 \leq i_1 < i_2 \leq n$, such that* $\alpha_{i_1} > 0, \alpha_{i_2} > 0$ *and: $\alpha = 0$ or $\alpha = \alpha_1 + \cdots + \alpha_n$.*

The proof is straightforward.

Proof of Theorem 1: Consider the finite dimensional subspace of E generated by the elements $x_1, \ldots, x_N$. The definition of the quasihypermetric property implies that this subspace is quasihypermetric too. Now since each finite dimensional real Banach space is isometrically isomorphic to some $(\mathbb{R}^n, \|.\|)$ ($n \geq 1$, $\|.\|$ a norm on $\mathbb{R}^n$) and of course the quasihypermetric property is invariant under isometries, it remains to show that:

Let $n \geq 1$ and $\|.\|$ be a quasihypermetric norm on $\mathbb{R}^n$. Furthermore let $N \geq 1$ and $y_1, \ldots, y_N$ be elements in $\mathbb{R}^n$. Then we have

$$\frac{1}{2N} \sum_{i=1}^{N} \|y_i - x\| + \|y_i + x\| \leq \frac{1}{N} \sum_{i=1}^{N} \|y_i\| + \frac{N-1}{N} \|x\|,$$

for all x in $P(y_1, \ldots, y_N) \subseteq \mathbb{R}^n$ defined as

$$P(y_1, \ldots, y_N) = \left\{ \sum_{i=1}^{N} \alpha_i y_i \text{ , such that } \alpha_1, \ldots, \alpha_N \in \mathbb{R} \text{ and } \max_{1 \leq i \leq N} |\alpha_i| \leq 1 \right\}.$$

By Theorem B (3) there exists some μ in $M^+(\Omega_{n-1})$ on the Euclidean unit sphere Ω_{n-1} of $\mathbb{R}^n$ such that

$$\|x\| = \int_{\Omega_{n-1}} |(x|y)| d\mu(y), \quad \text{for all } x \in \mathbb{R}^n.$$

Now let $x \in P(y_1, \ldots, y_N)$ and $y \in \Omega_{n-1}$. Since $x \in P(y_1, \ldots, y_N)$ we can find some $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ with $\max_{1 \leq i \leq N} |\alpha_i| \leq 1$ such that $x = \sum_{i=1}^{N} \alpha_i y_i$. Hence we get

$$|(x|y)| = \left| \sum_{i=1}^{N} \alpha_i (y_i|y) \right| \leq \sum_{i=1}^{N} |(y_i|y)|.$$

Take $\alpha = |(x|y)|$ and $\alpha_i = |(y_i|y)|$ for $1 \leq i \leq N$. Lemma 1 implies that

$$\frac{1}{N} \sum_{i=1}^{N} \max(|(y_i|y)|, |(x|y)|) \leq \frac{1}{N} \sum_{i=1}^{N} |(y_i|y)| + \frac{N-1}{N} |(x|y)|.$$

Therefore

$$\frac{1}{2N} \sum_{i=1}^{N} |(y_i - x|y)| + |(y_i + x|y)| \leq \frac{1}{N} \sum_{i=1}^{N} |(y_i|y)| + \frac{N-1}{N} |(x|y)|,$$

for all y in Ω_{n-1}.

Integration over Ω_{n-1} with respect to μ finishes the proof. ∎

Proof of Theorem 2: As mentioned in the proof of Theorem 1 we can assume that $a_1, \ldots, a_n$ is some Auerbach basis of the space $(\mathbb{R}^n, \|.\|)$ ($n \geq 1$ and $\|.\|$ a quasihypermetric norm on $\mathbb{R}^n$). Since $a_1, \dots, a_n$ is an Auerbach basis of $\mathbb{R}^n$ the subset $P(a_1, \dots, a_n)$ defined in Theorem 1 fulfills the relation

$$P(a_1, \dots, a_n) \supseteq \{x \in \mathbb{R}^n,\ \|x\| \leq 1\}.$$

Of course $\|a_1\| = \cdots = \|a_n\| = 1$ and hence the result follows by Theorem 1. ∎

Before showing Theorem 3, we prove the following lemmata:

LEMMA 2: *Let $n \geq 2$. Then we have*

$$\frac{1}{2n}\sum_{i=1}^{n} \|x - e_i\| + \|x + e_i\| = 1 + \frac{n-1}{n}\|x\|, \tag{1}$$

for all x in $l^1(n)$ with $\|x\|_\infty \leq 1$.

$$\frac{1}{2}(\|x - x_0\| + \|x + x_0\|) \leq 2 - \frac{1}{n}, \tag{2}$$

for all x in $l^1(n)$ with $\|x\| = 1$, where x_0 in $l^1(n)$ is defined as $x_0 = \left(\frac{1}{n}, \dots, \frac{1}{n}\right)$.
Moreover, equality holds if and only if $x \in \{e_1, \dots, e_n, -e_1, \dots, -e_n\}$.

The proof is straightforward.

LEMMA 3: *Let ν be a probability measure on the unit sphere $S_{l^1(n)}$ of $l^1(n)$ ($n \geq 2$) and assume that*

$$\int_{S_{l^1(n)}} \|x - y\| d\nu(y) \geq 2 - \frac{1}{n}$$

for all x in $S_{l^1(n)}$. Then we have

$$\nu = \frac{1}{2n}\sum_{i=1}^{n} \delta_{e_i} + \delta_{-e_i},$$

where $\delta_{e_i}(\delta_{-e_i})$ denotes the point measure on the i-th canonical vector $e_i(-e_i)$.

Proof: Let $x_0 = (1/n, \dots, 1/n)$. By assumption we get

$$\int_{S_{l^1(n)}} \frac{\|x_0 - y\| + \|x_0 + y\|}{2} d\nu(y) \geq 2 - \frac{1}{n}.$$

Applying Lemma 2, (2) we obtain

$$\text{supp}(\nu) \subseteq \{e_1, \dots, e_n, -e_1, \dots, -e_n\},$$

where supp(ν) denotes the support of ν.

Hence there are $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \geq 0$, $\sum_{i=1}^n \alpha_i + \beta_i = 1$, such that

$$\nu = \sum_{i=1}^n \alpha_i \delta_{e_i} + \beta_i \delta_{-e_i}.$$

The assumption again implies

$$\sum_{i=1}^n \alpha_i \|e_k - e_i\| + \beta_i \|e_k + e_i\| \geq 2 - \frac{1}{n},$$

for all $1 \leq k \leq n$.

Hence $\alpha_i \leq 1/2n$ and $\beta_i \leq 1/2n$, for all $1 \leq i \leq n$, and therefore

$$\alpha_1 = \cdots = \alpha_n = \beta_1 = \cdots = \beta_n = \frac{1}{2n}. \qquad \blacksquare$$

LEMMA 4: *Let E be a real n-dimensional quasihypermetric Banach space $(n \geq 2)$. Define*

$$N(E) = \max\{k \geq 2,\ \exists\, x_1, \ldots, x_k \in S_E \text{ with } \|x_i - x_j\| = \|x_i + x_j\| = 2,$$
$$\text{for all } 1 \leq i \neq j \leq k\}.$$

Then we have $N(E) \leq n$ and equality holds if and only if E is isometrically isomorphic to $l^1(n)$.

Proof: By Theorem B, (2) we can assume that E is an n-dimensional subspace of $L^1[0,1]$. For arbitrary real numbers α, β we have

$$|\alpha + \beta| + |\alpha - \beta| \leq 2(|\alpha| + |\beta|),$$

and equality holds if and only if $\alpha\beta = 0$. Integration of this inequality shows that for every $f, g \in L^1[0,1]$,

$$\|f + g\| + \|f - g\| \leq 2(\|f\| + \|g\|),$$

and equality holds if and only if f, g are disjointly supported, up to a set of measure zero. Therefore, if f, g are unit vectors satisfying $\|f + g\| = \|f - g\| = 2$ they are disjointly supported (up to a set of measure zero) and, in particular, linearly independent. Hence we get $N(E) \leq n$. In the case $N(E) = n$, E contains n disjointly supported functions (up to a set of measure zero) and therefore E

is isometrically isomorphic to $l^1(n)$. Of course $N(l^1(n)) = n$ and hence we are done.

Proof of Theorem 3: Let E be a real n-dimensional quasihypermetric Banach space ($n \geq 2$). Choose some Auerbach basis $a_1, \ldots, a_n$ of E (such a basis always exists, a proof is given, for example, in [27]).

Theorem 2 implies that

$$\frac{1}{2n}\sum_{i=1}^{n} \|x - a_i\| + \|x + a_i\| \leq 2 - \frac{1}{n},$$

for all x in S_E. Now Gross's Theorem (see Theorem A in Section 1) guarantees the existence of some x_0 in S_E, such that

$$r(E) = \frac{1}{2n}\sum_{i=1}^{n} \|x_0 - a_i\| + \|x_0 + a_i\|,$$

and hence $r(E) \leq 2 - 1/n$.

Lemma 2, (1) implies that

$$\frac{1}{2n}\sum_{i=1}^{n} \|e_i - x\| + \|e_i + x\| = 2 - \frac{1}{n},$$

for all x in $S_{l^1(n)}$ and therefore Gross's Theorem again leads to

$$r(l^1(n)) = 2 - \frac{1}{n}.$$

Now assume that $r(E) = 2 - 1/n$. It remains to show that E is isometrically isomorphic to $l^1(n)$.

We continue by induction on n.

$n = 2:$

In [28] it was shown that $r(E) = \frac{3}{2}$ if and only if the two dimensional real Banach space E is isometrically isomorphic to $l^1(2)$.

$n - 1 \mapsto n:$

Choose some Auerbach basis $a_1, \ldots, a_n$ of E and define the norm $\|.\|_0$ on $\mathbb{R}^n$ as

$$\|(x_1, \ldots, x_n)\|_0 = \|x_1 a_1 + \cdots + x_n a_n\|, \quad x_1, \ldots, x_n \in \mathbb{R};$$

then T: $E \to \mathbb{R}^n$, $x_1 a_1 + \cdots, +x_n a_n \mapsto (x_1, \ldots, x_n)$ becomes an isometric isomorphism from E to $(\mathbb{R}^n, \|.\|_0)$.

Since the definition of $r(E)$ is invariant under isometric isomorphisms, we have the following situation (for convenience replace $\|.\|_0$ by $\|.\|$):

Let $n \geq 3$ and $\|.\|$ be a quasihypermetric norm on $\mathbb{R}^n$ such that

$$\|x\|_\infty \leq \|x\| \leq \|x\|_1,$$

for all x in $\mathbb{R}^n$ (the canonical basis $e_1, \ldots, e_n$ forms an Auerbach basis of $(\mathbb{R}^n, \|.\|)$) and assume

$$r((\mathbb{R}^n, \|.\|)) = 2 - \frac{1}{n}.$$

We have to show that $(\mathbb{R}^n, \|.\|)$ is isometrically isomorphic to $l^1(n)$.

By Theorem B, (3) there exists a unique even measure μ in $M^+(\Omega_{n-1})$ on the Euclidean sphere Ω_{n-1} of $\mathbb{R}^n$ such that

$$\|x\| = \int_{\Omega_{n-1}} |(x|y)| d\mu(y), \quad \text{for all } x \text{ in } \mathbb{R}^n.$$

For short let $S = \{x \in \mathbb{R}^n,\ \|x\| = 1\}$ be the unit sphere of $(\mathbb{R}^n, \|.\|)$. Furthermore, define $f\colon S \to \mathbb{R}$ as

$$f(x) = \frac{1}{2n} \sum_{i=1}^{n} \|x - e_i\| + \|x + e_i\|, \quad \text{for all } x \text{ in } S$$

and let

$$A \subseteq S, \quad A = \{x \in S,\ f(x) = 2 - 1/n\}.$$

By Gross's Theorem and $r((\mathbb{R}^n, \|.\|)) = 2 - \frac{1}{n}$ it follows that A is a non-empty compact subset of S.

Now fix some y_0 in $\operatorname{supp}(\mu) \setminus \{e_1, \ldots, e_n, -e_1, \ldots, -e_n\}$ ($\operatorname{supp}(\mu)$ denotes the support of μ) and let a be an element of A.

Lemma 1 implies

$$\frac{1}{n} \sum_{i=1}^{n} \max(|(a|y)|, |(e_i|y)|) \leq \frac{1}{n} \sum_{i=1}^{n} |(e_i|y)| + \frac{n-1}{n} |(a|y)|,$$

for all y in Ω_{n-1}.

Since $f(a) = 2 - \frac{1}{n}$, we get

$$\int_{\Omega_{n-1}} \left[\frac{1}{n} \sum_{i=1}^{n} |(e_i|y)| + \frac{n-1}{n} |(a|y)| - \frac{1}{n} \sum_{i=1}^{n} \max(|(a|y)|, |(e_i|y)|) \right] d\mu(y) = 0.$$

Hence Lemma 1 again leads to

$$(a|y_0) = 0 \quad \text{or} \quad |(a|y_0)| = \sum_{i=1}^{n} |(e_i|y_0)|.$$

In the case $|(a|y_0)| = \sum_{i=1}^n |(e_i|y_0)| = \|y_0\|_1$ we obtain:

Since $y_0 \notin \{e_1, \dots, e_n, -e_1, \dots, -e_n\}$ choose $i_1, i_2 \in \{1, \dots, n\}$ as small as possible, such that $i_1 < i_2$, $(y_0|e_{i_1}) \neq 0$ and $(y_0|e_{i_2}) \neq 0$.

Now $\|a\|_\infty \leq \|a\| = 1$ implies

$$((a|e_{i_1}), (a|e_{i_2})) = (\operatorname{sgn}(y_0|e_{i_1}), \operatorname{sgn}(y_0|e_{i_2}))$$

or

$$((a|e_{i_1}), (a|e_{i_2})) = -(\operatorname{sgn}(y_0|e_{i_1}), \operatorname{sgn}(y_0|e_{i_2}))$$

and hence $\|a\|_1 \geq 2$.

Let $z(y_0) \in \mathbb{R}^n$ be defined as

$$z(y_0) = \operatorname{sgn}(y_0|e_{i_2})e_{i_1} - \operatorname{sgn}(y_0|e_{i_1})e_{i_2};$$

then

$$z(y_0) \neq 0 \quad \text{and} \quad (z(y_0)|a) = 0.$$

For short let $C = \operatorname{supp}(\mu) \setminus \{e_1, \dots, e_n, -e_1, \dots, -e_n\}$. Corresponding to y_0 in C we define

$$\begin{aligned} H_{y_0} &= ([y_0]^\perp, \|.\|), \\ H_{z(y_0)} &= ([z(y_0)]^\perp, \|.\|). \end{aligned}$$

H_{y_0} and $H_{z(y_0)}$ are both $(n-1)$-dimensional real quasihypermetric subspaces of $(\mathbb{R}^n, \|.\|)$. For short let S_{y_0} be the unit sphere of H_{y_0} and $S_{z(y_0)}$ be the unit sphere of $H_{z(y_0)}$.

Summing up we have the following situation:

Each y in C leads to subsets $A_1(y)$ and $A_2(y)$ of A such that

$$\begin{aligned} A_1(y) \cup A_2(y) &= A, \\ A_1(y) \cap A_2(y) &= \emptyset, \end{aligned}$$

$A_1(y) = \{a \in A, (a|y) = 0\}$ is a compact subset of the unit sphere S_y of H_y, $A_2(y) = \{a \in A, |(a|y)| = \|y\|_1\}$ is a compact subset of the unit sphere $S_{z(y)}$ of $H_{z(y)}$ and

$$\|a\|_1 \geq 2 \quad \text{for all } a \text{ in } A_2(y).$$

Now we consider two cases:

(1) $C = \emptyset$

This implies

$$\|x\| = \sum_{i=1}^n \gamma_i |(x|e_i)| \quad \text{for some } \gamma_1, \dots, \gamma_n \geq 0.$$

Since $\|e_i\| = 1$, for all $1 \leq i \leq n$, we get $\|x\| = \sum_{i=1}^{n} |(x|e_i)| = \|x\|_1$ and we are done.

(2) $C \neq \emptyset$

Since $r((\mathbb{R}^n, \|.\|)) = 2 - 1/n$, Remark 1(c) implies the existence of some ν in $M^1(S)$, such that

$$\int_S \|x-u\| d\nu(u) \geq 2 - \frac{1}{n}, \quad \text{for all } x \text{ in } S.$$

Hence

$$\int_S f(u) d\nu(u) \geq 2 - \frac{1}{n},$$

but $f(u) \leq 2 - \frac{1}{n}$ for all u in S and therefore we get

$$\operatorname{supp}(\nu) \subseteq A.$$

FIRST STEP: We claim that $0 < \nu(A_1(y)) < 1$ for all y in C.

Assume that $\nu(A_1(y_0)) = 0$, for some y_0 in C. Since

$$\int_S \|x-u\| d\nu(u) = \int_A \|x-u\| d\nu(u) = \int_{A_2(y_0)} \|x-u\| d\nu(u),$$

for all x in S, we get

$$2 - \frac{1}{n} \leq \int_{A_2(y_0)} \|x-u\| d\nu(u),$$

for all x in S.

Now consider ν as an element of $M^1(S_{z(y_0)})$. As shown before, we know that

$$r(H_{z(y_0)}) \leq 2 - \frac{1}{n-1}$$

and hence Gross's Theorem (see Remark 1(c)) implies some x_0 in $S_{z(y_0)}$, such that

$$\int_{A_2(y_0)} \|x_0 - u\| d\nu(u) \leq 2 - \frac{1}{n-1},$$

a contradiction to $x_0 \in S_{z(y_0)} \subseteq S$.

The case $\nu(A_1(y_0)) = 1$ is treated as the case $\nu(A_1(y_0)) = 0$ (consider ν as an element of $M^1(S_{y_0})$, $A_1(y_0) \subseteq S_{y_0}$). Hence $0 < \nu(A_1(y)) < 1$ for all y in C.

Choose some y in C. Then we have

$$\begin{aligned}\int_A \|x-u\| d\nu(u) &= \int_{A_1(y)} \|x-u\| d\nu(u) + \int_{A_2(y)} \|x-u\| d\nu(u) \\ &= \nu(A_1(y)) \int_{A_1(y)} \|x-u\| d\frac{\nu}{\nu(A_1(y))}(u) \\ &\quad + \nu(A_2(y)) \int_{A_2(y)} \|x-u\| d\frac{\nu}{\nu(A_2(y))}(u),\end{aligned}$$

for all x in S.

Now let $\nu_y \in M^1(S_y)$ be $\nu_y = \nu/\nu(A_1(y))$ on S_y and $\nu_{z(y)} \in M^1(S_{z(y)})$ be $\nu_{z(y)} = \nu/\nu(A_2(y))$ on $S_{z(y)}$. Hence we obtain the formula

$$(\star) \quad 2 - \frac{1}{n} \le \nu(A_1(y)) \int_{S_y} \|x-u\| d\nu_y(u) + \nu(A_2(y)) \int_{S_{z(y)}} \|x-u\| d\nu_{z(y)}(u),$$

for all x in S and all y in C.

SECOND STEP: We claim that $1/n \le \nu(A_1(y)) \le (n-1)/n$, for all y in C.

Fix some y_0 in C. As shown before, we know that

$$r(H_{y_0}) \le 2 - \frac{1}{n-1} \quad \text{and} \quad r(H_{z(y_0)}) \le 2 - \frac{1}{n-1}$$

and hence Remark 1(c) and Gross's Theorem imply some x_1 in S_{y_0} and x_2 in $S_{z(y_0)}$ such that

$$\int_{S_{y_0}} \|x_1 - u\| d\nu_{y_0}(u) \le 2 - \frac{1}{n-1} \quad \text{and}$$
$$\int_{S_{z(y_0)}} \|x_2 - u\| d\nu_{z(y_0)}(u) \le 2 - \frac{1}{n-1}.$$

Therefore formula $(\star)$ yields

$$2 - \frac{1}{n} \le \nu(A_1(y_0)) \left(2 - \frac{1}{n-1}\right) + \nu(A_2(y_0)) \cdot 2$$

and

$$2 - \frac{1}{n} \le \nu(A_1(y_0)) \cdot 2 + \nu(A_2(y_0)) \left(2 - \frac{1}{n-1}\right).$$

From this we obtain

$$\frac{1}{n} \le \nu(A_1(y_0)) \le \frac{n-1}{n}.$$

Now let $\gamma_i = 2\mu\{e_i\}$ $(= 2\mu\{-e_i\})$, for all $1 \le i \le n$. We get

$$\|x\| = \sum_{i=1}^{n} \gamma_i |(x|e_i)| + \int_C |(x|y)| d\mu(y),$$

for all x in $\mathbb{R}^n$.

Furthermore, define b_k as

$$b_k = \int_A \frac{\|e_k - a\| + \|e_k + a\|}{2} d\nu(a),$$

for all $1 \le k \le n$.

We expect to prove that μ is supported on precisely $2n$ points and since $C \ne \emptyset$ we continue:

THIRD STEP: We claim the existence of some $1 \leq i_0 \leq n$, such that $\gamma_{i_0} = 0$.

Assume $\gamma_1, \ldots, \gamma_n > 0$: By definition of b_k we have

$$b_k = \int_A \sum_{i=1}^n \gamma_i \max(|(e_i|e_k)|, |(e_i|a)|) d\nu(a) + \int_A \int_C \max(|(e_k|y)|, |(a|y)|) d\mu(y) d\nu(a).$$

Hence

$$\begin{aligned} b_k = & \int_A \sum_{i=1}^n \gamma_i |(e_i|a)| d\nu(a) - \gamma_k \int_A |(e_k|a)| d\nu(a) \\ & + \gamma_k + \int_C \int_{A_1(y)} \max(|(e_k|y)|, |(a|y)|) d\nu(a) d\mu(y) \\ & + \int_C \int_{A_2(y)} \max(|(e_k|y)|, |(a|y)|) d\nu(a) d\mu(y). \end{aligned}$$

Since

$$1 = \|a\| = \sum_{i=1}^n \gamma_i |(e_i|a)| + \int_C |(a|y)| d\mu(y),$$

for all a in A, and recalling the definition of the subsets $A_1(y)$, $A_2(y)$ of A, we obtain

$$\begin{aligned} b_k = 1 & - \int_A \int_C |(a|y)| d\mu(y) d\nu(a) - \gamma_k \int_A |(e_k|a)| d\nu(a) \\ & + \gamma_k + \int_C \nu(A_1(y)) |(e_k|y)| d\mu(y) \\ & + \int_C \nu(A_2(y)) \|y\|_1 d\mu(y). \end{aligned}$$

Now

$$\begin{aligned} \int_A \int_C |(a|y)| d\mu(y) d\nu(a) & = \int_C \int_{A_2(y)} \|y\|_1 d\nu(a) d\mu(y) \\ & = \int_C \nu(A_2(y)) \|y\|_1 d\mu(y). \end{aligned}$$

Therefore

$$b_k = 1 + \gamma_k - \gamma_k \int_A |(e_k|a)| d\nu(a) + \int_C \nu(A_1(y)) |(e_k|y)| d\mu(y),$$

for all $1 \leq k \leq n$.

Since $\|e_k\| = \|-e_k\| = 1$ and

$$b_k = \frac{1}{2} \left(\int_S \|e_k - a\| d\nu(a) + \int_S \|e_k + a\| d\nu(a) \right),$$

we have

$$b_k \geq 2 - \frac{1}{n}, \quad \text{for all } 1 \leq k \leq n.$$

As shown before, we know that

$$\nu(A_1(y)) \leq \frac{n-1}{n},$$

for all y in C, and hence

$$2 - \frac{1}{n} \leq b_k \leq 1 + \gamma_k - \gamma_k \int_A |(e_k|a)| d\nu(a) + \frac{n-1}{n} \int_C |(e_k|y)| d\mu(y),$$

for all $1 \leq k \leq n$.

Since

$$\|e_k\| = 1 = \gamma_k + \int_C |(e_k|y)| d\mu(y),$$

for all $1 \leq k \leq n$, we get

$$\gamma_k \left(\frac{1}{n} - \int_A |(e_k|a)| d\nu(a) \right) \geq 0,$$

for all $1 \leq k \leq n$.

Now by assumption $\gamma_1, \ldots, \gamma_n > 0$ and hence, by summation, we obtain

$$1 \geq \int_A \sum_{k=1}^{n} |(e_k|a)| d\nu(a) = \int_A \|a\|_1 d\nu(a).$$

Choose some y_0 in C:

$$\begin{aligned} 1 \geq \int_A \|a\|_1 d\nu(a) &= \int_{A_1(y_0)} \|a\|_1 d\nu(a) + \int_{A_2(y_0)} \|a\|_1 d\nu(a) \\ &\geq \int_{A_1(y_0)} \|a\| d\nu(a) + \int_{A_2(y_0)} 2 d\nu(a) = \nu(A_1(y_0)) + 2\nu(A_2(y_0)). \end{aligned}$$

Hence $\nu(A_2(y_0)) = 0$, a contradiction to $\nu(A_2(y_0)) \geq 1/n$.

FOURTH STEP: We claim the existence of some $\bar{y}$ in C, such that $\nu(A_1(\bar{y})) = (n-1)/n$.

Remember we have found some $1 \leq i_0 \leq n$ such that $\gamma_{i_0} = 0$. Hence

$$1 = \|e_{i_0}\| = \int_C |(e_{i_0}|y)| d\mu(y) \quad \text{and} \quad b_{i_0} = 1 + \int_C \nu(A_1(y)) |(e_{i_0}|y)| d\mu(y).$$

Since $b_{i_0} \geq 2 - \frac{1}{n}$, we get

$$\int_C \left[\frac{n-1}{n} - \nu(A_1(y)) \right] |(e_{i_0}|y)| d\mu(y) \leq 0.$$

Now

$$\int_A |(a|y)|d\nu(a) = \nu(A_2(y))\|y\|_1,$$

for all y in C, and therefore the function

$$y \mapsto \nu(A_1(y)) = 1 - \frac{1}{\|y\|_1}\int_A |(a|y)|d\nu(a)$$

is continuous on C.

Since $(n-1)/n \geq \nu(A_1(y))$ for all y in C, we obtain

$$\left[\frac{n-1}{n} - \nu(A_1(y))\right] |(e_{i_0}|y)| = 0,$$

for all y in C.

But

$$1 = \|e_{i_0}\| = \int_C |(e_{i_0}|y)|d\mu(y)$$

implies some $\bar{y}$ in C, such that $(e_{i_0}|\bar{y}) \neq 0$. It follows that

$$\nu(A_1(\bar{y})) = \frac{n-1}{n}.$$

FIFTH STEP: We finish the proof.

Now take $\bar{y}$ in C as found before ($\nu(A_1(\bar{y})) = (n-1)/n$) and apply formula ($\star$):

$$2 - \frac{1}{n} \leq \frac{n-1}{n}\int_{S_{\bar{y}}} \|x-u\|d\nu_{\bar{y}}(u) + \frac{1}{n}\int_{S_{z(\bar{y})}} \|x-u\|d\nu_{z(\bar{y})}(u),$$

for all x in S.

Assume that

$$\int_{S_{\bar{y}}} \|x_0 - u\|d\nu_{\bar{y}}(u) < 2 - \frac{1}{n-1},$$

for some x_0 in $S_{\bar{y}} \subseteq S$. This would imply

$$2 - \frac{1}{n} < \frac{n-1}{n}\left(2 - \frac{1}{n-1}\right) + \frac{1}{n}\cdot 2 = 2 - \frac{1}{n},$$

a contradiction.

Therefore we get

$$\int_{S_{\bar{y}}} \|x-u\|d\nu_{\bar{y}}(u) \geq 2 - \frac{1}{n-1},$$

for all x in $S_{\bar{y}}$.

Now $r(H_{\bar{y}}) \leq 2 - 1/(n-1)$ and Remark 1(c) imply

$$r(H_{\bar{y}}) = 2 - \frac{1}{n-1}.$$

By the induction hypothesis we obtain that $H_{\bar{y}}$ is isometrically isomorphic to $l^1(n-1)$.

As shown above

$$\int_{S_{\bar{y}}} \|x-u\| d\nu_{\bar{y}}(u) \geq 2 - \frac{1}{n-1},$$

for all x in $S_{\bar{y}}$, and since $H_{\bar{y}}$ is isometrically isomorphic to $l^1(n-1)$ we can apply Lemma 3 and obtain

$$\int_{S_{\bar{y}}} \|x-u\| d\nu_{\bar{y}}(u) = \frac{1}{2(n-1)} \sum_{i=1}^{n-1} \|f_i - x\| + \|f_i + x\|,$$

for all x in $S_{\bar{y}}$, where $\{f_1, \ldots, f_{n-1}, -f_1, \ldots, -f_{n-1}\}$ is the set of extreme points of the closed unit ball $\{x \in H_{\bar{y}}\,,\ \|x\| \leq 1\}$ of $H_{\bar{y}}$.

By Lemma 2, (1) we have

$$\frac{1}{2(n-1)} \sum_{i=1}^{n-1} \|f_i - x\| + \|f_i + x\| = 2 - \frac{1}{n-1},$$

for all x in $S_{\bar{y}}$.

Hence by formula ($\star$) we get

$$2 - \frac{1}{n} \leq \frac{n-1}{n}\left(2 - \frac{1}{n-1}\right) + \frac{1}{n}\int_{S_{z(\bar{y})}} \|x-u\| d\nu_{z(\bar{y})}(u),$$

for all x in $S_{\bar{y}}$.

Therefore

$$\int_{S_{z(\bar{y})}} \|x-u\| d\nu_{z(\bar{y})}(u) = 2,$$

for all x in $S_{\bar{y}}$.

For $x = f_1, \ldots, f_{n-1}, -f_1, \ldots, -f_{n-1}$ and by summation we get

$$\int_{S_{z(\bar{y})}} \left[\frac{1}{2(n-1)} \sum_{i=1}^{n-1} \|f_i - u\| + \|f_i + u\| - 2\right] d\nu_{z(\bar{y})}(u) = 0.$$

Hence there exists some f_0 in $S_{z(\bar{y})}$ such that

$$\frac{1}{2(n-1)} \sum_{i=1}^{n-1} \|f_i - f_0\| + \|f_i + f_0\| = 2.$$

Therefore $\|f_0 - f_i\| = \|f_0 + f_i\| = 2$, for all $1 \leq i \leq n-1$.

Since $f_1, \ldots, f_{n-1}, -f_1, \ldots, -f_{n-1}$ are the extreme points of the closed unit ball of a subspace isometrically isomorphic to $l^1(n-1)$ (namely $H_{\bar{y}}$), we have of course $\|f_i - f_j\| = \|f_i + f_j\| = 2$, for all $1 \leq i \neq j \leq n-1$.

Summing up we have

$$\|f_i\| = 1 \quad \text{and} \quad \|f_i - f_j\| = \|f_i + f_j\| = 2,$$

for all $0 \leq i \neq j \leq n-1$.

Finally, Lemma 4 implies that $(\mathbb{R}^n, \|.\|)$ is isometrically isomorphic to $l^1(n)$. ∎

We need the following lemmata:

LEMMA 5: *Let E be a two dimensional real Banach space.*

(1) *Let P be a convex polygon in E.*
Then $p(P)$ is given by the sum of lengths over all sides of P.

(2) *Let L and K be two compact convex subsets of E such that $L \subseteq K$.*
Then we have $p(L) \leq p(K)$.

(For the definition of the Minkowski perimeter $p(.)$ see Section 2 of this paper.) For a proof see Lemma 11.1 in [17].

LEMMA 6: *Let E be a two dimensional real Banach space and let $OE = \{x \in E, \|x\| \leq 1\}$ denote the closed unit ball of E. Then we have*

$$6 \leq p(OE) \leq 8$$

and $p(OE) = 8$ if and only if E is isometrically isomorphic to $l^1(2)$.

For a proof see Satz 11.9 in [17].

LEMMA 7: *Let E be a two dimensional real Banach space and let K be a compact convex subset of E. Then we have*

$$p(K) \leq 4D(K),$$

where $p(K)$ denotes the Minkowski perimeter and $D(K)$ the diameter of K.

The proof is essentially the same as of Satz 11.9 ($p(OE) \leq 8$) given in [17]:

Choose some Auerbach basis a_1, a_2 of E. Let g_1, g_2 resp. h_1, h_2 be the supporting lines of K with direction given by a_1 resp. a_2. Choose points $x_i \in K \cap g_i$ and $y_i \in K \cap h_i$, for $1 \leq i \leq 2$.

Furthermore, define the points z_{ij} as $z_{ij} = g_i \cap h_j$, for $1 \leq i, j \leq 2$.

Since a_1, a_2 forms an Auerbach basis of E, we get

$$D(K) \geq \|x_1 - x_2\| \geq \|z_{11} - z_{21}\| = \|z_{12} - z_{22}\|$$

and

$$D(K) \geq \|y_1 - y_2\| \geq \|z_{11} - z_{12}\| = \|z_{21} - z_{22}\|.$$

Lemma 5 implies

$$p(K) \leq \|z_{11} - z_{21}\| + \|z_{12} - z_{22}\| + \|z_{11} - z_{12}\| + \|z_{21} - z_{22}\|,$$

and the result follows. ∎

Proof of Theorem 4: The proof is based on an idea given by L. Fejes Tóth in Theorem 1 of [10].

Let $N \geq 1$ and $x_1, \dots, x_N$ be points in K. Assume that x_1 does not lie on the boundary of K:

Since the function $x \mapsto \sum_{i=2}^N \|x - x_i\|$ is convex on K, we can find some extreme point e on the boundary of K such that

$$\sum_{i=2}^N \|e - x_i\| \geq \sum_{i=2}^N \|x_1 - x_i\|$$

and hence the value of $\sum_{i,j=1}^N \|x_i - x_j\|$ increases, if x_1 is replaced by e. Repeating this process we can assume that all given points lie on the boundary of K and w.l.o.g. let $x_1, x_2, \dots, x_N$ be the cyclical order of the given point set $\{x_1, \dots, x_N\}$.

Now define

$$x_{N+1} = x_1, \quad x_{N+2} = x_2, \quad \dots, \quad x_{2N-1} = x_{N-1}$$

and let

$$S_k = \sum_{i=1}^N \|x_i - x_{i+k}\|,$$

for all $1 \leq k \leq \lfloor \frac{N}{2} \rfloor$.

Furthermore, let $A(i,k)$ denote the closed arc on the boundary of K joining x_i and x_{i+k} (go in this direction: $x_i \to x_{i+1} \to \cdots \to x_{i+k}$) and let $\delta(i,k)$ be the length of $A(i,k)$ ($\delta(i,k) + \|x_i - x_{i+k}\|$ is equal to the Minkowski perimeter of the convex hull of $A(i,k)$).

Now we have

$$\sum_{i=1}^N \delta(i,k) = kp(K),$$

for all $1 \le k \le \lfloor \frac{N}{2} \rfloor$, and since the triangle inequality forces $\delta(i,k) \ge \|x_i - x_{i+k}\|$, we obtain

$$S_k \le kp(K),$$

for all $1 \le k \le \lfloor \frac{N}{2} \rfloor$.

Of course, we have

$$S_k \le ND(K),$$

for all $1 \le k \le \lfloor \frac{N}{2} \rfloor$ and

$$\sum_{i,j=1}^{N} \|x_i - x_j\| = \begin{cases} 2 \cdot \sum_{k=1}^{\lfloor \frac{N}{2} \rfloor - 1} S_k + S_{\lfloor \frac{N}{2} \rfloor}, & N \equiv 0(2), \\ 2 \cdot \sum_{k=1}^{\lfloor \frac{N}{2} \rfloor} S_k, & N \equiv 1(2). \end{cases}$$

For short let

$$\sigma_N = \frac{1}{N^2} \sum_{i,j=1}^{N} \|x_i - x_j\|.$$

Now

$$\sigma_N \le \frac{1}{N^2} \left(2 \cdot \sum_{k=1}^{\lfloor \frac{N}{4} \rfloor} kp(K) + 2 \cdot \sum_{k=\lfloor \frac{N}{4} \rfloor + 1}^{\lfloor \frac{N}{2} \rfloor - 1} ND(K) + ND(K) \right),$$

in the case $N \equiv 0(2)$ and

$$\sigma_N \le \frac{1}{N^2} \left(2 \cdot \sum_{k=1}^{\lfloor \frac{N}{4} \rfloor} kp(K) + 2 \cdot \sum_{k=\lfloor \frac{N}{4} \rfloor + 1}^{\lfloor \frac{N}{2} \rfloor} ND(K) \right),$$

in the case $N \equiv 1(2)$.

Let

$$\alpha(K) = \frac{D(K)}{2} + \frac{p(K)}{16},$$
$$\beta(K) = D(K) - \frac{p(K)}{4},$$

and note that $\beta(K) \ge 0$ by Lemma 7.

Routine calculations lead to

$$\sigma_N \leq \begin{cases} \alpha(K) - \frac{1}{N}\beta(K), & N \equiv 0\ (4), \\ \alpha(K) - \frac{1}{2N}\beta(K) - p(K)\frac{3}{16N^2}, & N \equiv 1\ (4), \\ \alpha(K) - \frac{p(K)}{4N^2}, & N \equiv 2\ (4), \\ \alpha(K) - \left[p(K)\frac{2N+3}{16N^2} - \frac{D(K)}{2N}\right], & N \equiv 3\ (4). \end{cases}$$

Hence in the cases $N \equiv 0, 1, 2\ (4)$ we are done.

But as before we also obtain

$$\sigma_N \leq \frac{1}{N^2}\left(2 \cdot \sum_{k=1}^{\lfloor \frac{N}{4} \rfloor + 1} kp(K) + 2 \cdot \sum_{k=\lfloor \frac{N}{4} \rfloor + 2}^{\lfloor \frac{N}{2} \rfloor} ND(K)\right),$$

in the case $N \equiv 1\ (2)$.

$N \equiv 3\ (4)$ yields

$$\sigma_N \leq \alpha(K) - \left[\frac{3D(K)}{2N} - p(K)\frac{6N+5}{16N^2}\right].$$

It is easy to see that

$$\max\left(p(K)\frac{2N+3}{16N^2} - \frac{D(K)}{2N}, \frac{3D(K)}{2N} - p(K)\frac{6N+5}{16N^2}\right) \geq 0,$$

for each $N \geq 1$, and therefore we have finished the proof. ∎

Proof of Corollary 1: As noted in the proof of Theorem 4, a simple convexity argument leads to

$$M(S_E, \|.\|) = M(OE, \|.\|).$$

Now apply Theorem 4 (of course $D(OE) = 2$). ∎

Proof of Corollary 2: Let $N \geq 1$ and $x_1, \ldots, x_N$ in $\mathbb{R}^2$ with $\|x_i - x_j\|_2 \leq 1$ for all $1 \leq i, j \leq N$. Let X be the convex hull of $\{x_1, \ldots, x_N\}$. Now it is well known that each compact convex subset Y of $\mathbb{R}^2$ is included in some compact convex subset $\tilde{Y}$ of $\mathbb{R}^2$ such that $D(Y) = D(\tilde{Y})$ and $\tilde{Y}$ is of constant width $D(Y)$.

Hence we can find some compact convex subset $\tilde{X}$ of constant width $D(X) = D(\tilde{X})$ with $X \subseteq \tilde{X}$.

By Barbier's Theorem (all compact convex subsets of the Euclidean plane of constant with λ have perimeter $\pi.\lambda$) we obtain $p(\tilde{X}) = \pi.D(\tilde{X}) = \pi.D(X)$. Since $D(X) \leq 1$ by assumption and applying Theorem 4 we get

$$\frac{1}{N^2}\sum_{i,j=1}^{N} \|x_i - x_j\| \leq M(X, \|.\|_2) \leq M(\tilde{X}, \|.\|_2) \leq \frac{1}{2} + \frac{\pi}{16},$$

and hence $\gamma(l^2(2)) \leq \frac{1}{2} + \frac{\pi}{16}$.

Now let X be a compact connected subset of $\mathbb{R}^2$ with $D(X) = 1$. As noted in Section 3 we have $r(X, \|.\|_2) \leq M(X, \|.\|_2)$ and therefore $k_2 \leq \gamma(l^2(2))$. ∎

Proof of Corollary 3: Let $N \geq 1$ and $x_1, \dots, x_N$ in E such that

$$\|x_i - x_j\| \leq 1$$

for all $1 \leq i, j \leq N$.

Let K be the convex hull of $x_1, \dots, x_N$. Of course we have $D(K) \leq 1$. Hence

$$\frac{1}{N^2}\sum_{i,j=1}^{N} \|x_i - x_j\| \leq \frac{1}{2} + \frac{p(K)}{16} \leq \frac{3}{4},$$

by Theorem 4 and Lemma 7. Therefore we get

$$\gamma(E) \leq \frac{3}{4}.$$

It remains to show that

$$\gamma(l^1(2)) = \frac{3}{4}.$$

Take $x_1 = (\frac{1}{2}, 0)$, $x_2 = (-\frac{1}{2}, 0)$, $x_3 = (0, \frac{1}{2})$ and $x_4 = (0, -\frac{1}{2})$. We have

$$\frac{1}{16}\sum_{i,j=1}^{4} \|x_i - x_j\|_1 = \frac{3}{4},$$

and hence we are done. ∎

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